"In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

THEORY AND APPLICATION OF EXPANSION, CONTRACTION, AND ISOMETRIC MAPPINGS

A THESIS

Presented to

the Faculty of the Graduate Division

by

Wilbur James Stiles

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

September, 1961

THEORY AND APPLICATION OF EXPANSION, CONTRACTION, AND ISOMETRIC MAPPINGS

Approved by:

Date of Approval:

September 9, 1961

ACKNOWLEDGEMENT

I wish to thank Dr. R. H. Kasriel, not only for suggesting this topic but for the guidance he has given me during its development. I am indebted to Dr. W. J. Kammerer and Dr. D. L. Finn for reading the manuscript and for offering valuable suggestions. I also want to thank Dr. J. A. Nohel and the National Science Foundation for a research grant during the summer of 1961.

TABLE OF CONTENTS

SUMMARY

This paper is a study of contraction, expansion, and isometric mappings and their applications to various fields of mathematics.

Let R be a complete metric space. If $A: R \to R$ is a contraction mapping, there is a unique element x in R such that $Ax = x$. The fixed point x is the limit of the sequence $[x_n]$, where x_1 is any element of R and $x_n = Ax_{n-1}$ for $n > 1$. This theorem, which will be proved in Chapter one, unifies many existence proofs in various fields of analysis. It also provides a direct method for finding fixed points which is particularly valuable in applied mathematics.

The expansion mapping is presented in Chapter Two as an extension of the concept of contraction mapping. The inverse of an expansion mapping exists and is a contraction mapping. Therefore, many of the results obtained for contraction mappings are valid for expansion mappings.

The notion of a local expansion mapping and a local contraction mapping is presented to deal with problems that cannot be solved by global contractions or expansions. Some results are obtained directly by using the local conditions. In other cases it is shown that a local contraction is sufficient to insure a global contraction.

The following theorem is proved in Chapter Three. Let f be a non-expanding mapping which can be represented as the uniform limit of a sequence of contraction mappings and let R be a compact metric space. If $f(R) < R$ there is at least one point x in R such that $f(x) = x$.

Included in Chapter Four is a proof, introduced by Freudenthal and Hurewicz in [4], which states that a non-expanding mapping of a totally-bounded metric space onto itself is an isometry.

The norm of a linear space and a linear function is defined in Chapter Five to facilitate the development of the succeeding chapters.

Most of the applications presented in Chapter Six through Chapter Ten utilize the fixed point theorem. In Chapter Six it is shown that there exists a unique probability vector x such that $Ax = x$ when A is a regular transition matrix for a Markov chain. In Chapter Seven the existence of a unique solution for a special system of linear equations is proved. Other fields of application included are systems of non-linear equations and integral equations.

CHAPTER I

CONTRACTION MAPPINGS

Definition 1-1. A metric space is a pair of objects: a set X, whose elements are called points, and a distance, i.e., a single-valued, nonnegative, real valued function $d(x,y)$, defined for arbitrary x and y in X and satisfying the following conditions:

$$d(x,y) = 0 \text{ if and only if } x = y$$
$$d(x,y) = d(y,x)$$
$$d(x,y) + d(y,z) \leq d(x,z)$$

The metric space will usually be denoted by $R = (X,d)$.

Definition 1-2. A mapping f of a metric space R into itself is said to be a contraction mapping if there exists a number $\alpha < 1$ such that $d(fx,fy) \leq \alpha\, d(x,y)$ for any two points x and y in R.

Definition 1-3. A metric space R is said to be complete if every fundamental (Cauchy) sequence in R converges to a point in R.

Theorem 1-4. Every contraction mapping f defined on a complete metric space R has one and only one fixed point.

Proof. Let x_0 be any element in R and consider the sequence $[x_n]$ where $x_1 = Ax_0$, $x_n = Ax_{n-1}$ for $n \geq 1$. Assume $n < m$. Then

$$\begin{aligned} d(x_n,x_m) &= d(f^n x_0, f^m x_0) \leq \alpha\, d(f^{n-1}x_0, f^{m-1}x_0) \\ &\leq \alpha^n\, d(x_0, f^{m-n}x_0) \end{aligned}$$

$$\leq \alpha^n [\, d(x_0,x_1) + d(x_1,x_2) + \ldots + d(x_{m-n-1},x_{m-n})\,]$$

$$\leq \alpha^n [\, d(x_0,x_1) + \alpha\, d(x_0,x_1) + \ldots + \alpha^{m-n-1}\, d(x_0,x_1)\,]$$

$$= \alpha^n\, d(x_0,x_1)\, [\, 1 + \alpha + \alpha^2 + \ldots + \alpha^{m-n-1}\,]$$

$$\leq \alpha^n\, d(x_0,x_1)\, [\, \frac{1}{1-\alpha}].$$

Since $0 \leq \alpha < 1$, $[x_n]$ is a fundamental sequence and, by the completeness of R, has a limit in R. Let $x = \lim x_n$. Continuity of f implies that $fx = f(\lim x_n) = \lim fx_n = \lim x_{n+1} = x$.

To show uniqueness, suppose there exists an x in R and a y in R such that $fx = x$ and $fy = y$ but $x \neq y$. Then

$$0 < d(x,y) = d(fx,fy) \leq \alpha\, d(x,y)$$

where $\alpha < 1$. This is clearly impossible. Thus x is unique.

<u>Theorem 1-5</u>. Let $f: R \to R$ be a continuous mapping and R a complete metric space. Let the mapping f^n be a contraction mapping for some positive integer n. Then the equation $fx = x$ has one and only one solution in R.

<u>Proof</u>. Let x be any element in R and consider the sequence $[x_{nk}]_{k=1}^{\infty}$, where $x_{nk} = f^{nk}x$. By a repetition of the argument used above, $[x_{nk}]_{k=1}^{\infty}$ is a fundamental sequence. Let $x_0 = \lim_k x_{nk}$. Then

$$fx_0 = f(\lim_k x_{nk}) = \lim_k fx_{nk} = \lim_k f^{kn}fx$$

and

$$d(f^{kn}fx, f^{kn}x) \leq \alpha\, d(f^{(k-1)n}fx, f^{(k-1)n}x) \leq \ldots \leq \alpha^k\, d(fx,x).$$

Hence $\lim_k d(f^{nk}fx, f^{nk}x) = 0$. So $\lim_k f^{nk}fx = \lim_k f^{nk}x = \lim_k x_{nk} = x_0$ and $f(\lim_k x_{nk}) = x_0$, i.e., $fx_0 = x_0$.

To show uniqueness, suppose that there exists an x and y in R such that $fx = x$ and $fy = y$ where $x \neq y$. Then $f^n x = x$, $f^n y = y$ and

$$0 < d(x,y) = d(f^{2n}x, f^{2n}y) \leq \alpha\, d(f^n x, f^n y) = \alpha\, d(x,y).$$

$\alpha < 1$ implies a contradiction and hence the fixed point must be unique.

Remark. The fact that f^n is a contraction mapping does not imply that f is continuous. This is seen by considering the function $f: E_2 \to E_2$ such that $fx = (0,0)$ for all x in E_2 except $x = (1,0)$, while $f(1,0) = (2,0)$. f^2 is a contraction mapping; however, f is not continuous.

Corollary 1-6. Let $f: R \to R$ be a continuous mapping of a complete metric space R. Let f^n be a contraction mapping for some integer $n \geq 1$. Then the fixed point x_0 of f is the limit of the sequence $[x_n]$, where x_1 is any element of R and $x_n = fx_{n-1}$ for $n \geq 1$.

Proof. $d(f^{kn}f^p x, f^{kn}x) \leq \alpha\, d(f^{(k-1)n}f^p x, f^{(k-1)n}x) \leq \ldots$

$$\leq \alpha^k\, d(f^p x, x)$$

for any x in R, $k = 1, 2, \ldots$ and $p = 1, 2, \ldots, n-1$. Hence $\lim_k d(f^{kn}f^p x, f^{kn}x) = 0$, i.e., $\lim_k f^{kn}f^p x = \lim_k f^{kn}x$ or $\lim_k f^{kn+p}x = x_0$. Because this relation holds for each p such that $p = 1, 2, \ldots, n-1$, $\lim f^n x = x_0$ for each x in R as required.

Definition 1-7. A mapping f of a metric space R into itself is said to be a local contraction mapping if for every x in R there exists an $\varepsilon > 0$ and an $\alpha < 1$ such that $d(fy, fz) \leq \alpha\, d(y,z)$ whenever y and z are in $N(x, \varepsilon)$.

Definition 1-8. A mapping f of metric space R into itself is said to be (ε,α)-uniformly locally contractive if f is locally contractive and if the choice of ε and α does not depend on x.

Theorem 1-9. Let $f:R \to R$ be an (ε,α)-uniformly locally contractive mapping and R a convex subset of a normed linear space. Then f is a contraction mapping on R.

Proof. Let r and s be any elements in R. Since R is a convex set, there is a line $L(r,s)$ contained in R connecting r and s. Let $r = x_0, x_1, \ldots, x_n = s$ be a partition of $L(r,s)$ such that $d(x_{i-1},x_i) < \varepsilon$ for $1 \leq i \leq n$. Then

$$
\begin{aligned}
d(fr,fs) &\leq d(fr,fx_1) + d(fx_1 fx_2) + \ldots + d(fx_{n-1},fs) \\
&\leq \alpha\,[\, d(r,x_1) + \ldots + d(x_{n-1},s)\,] \\
&= \alpha\, d(r,s).
\end{aligned}
$$

Remark. Theorem 1-9 is given by Edelstein in [1] for a complete, convex metric space.

Definition 1-10. A metric space R is said to be ε-chainable if for every two elements a and b in R, there exists a finite number of points, $a = x_0, x_1, \ldots, x_n = b$, such that $d(x_{i-1},x_i) < \varepsilon$ for $1 \leq i \leq n$.

Theorem 1-11. Let R be a complete, ε-chainable, metric space and $f:R \to R$ an (ε,α)-uniformly locally contractive mapping. Then there exists a unique point x in R, such that $fx = x$.

Proof. Let x_0 be any element of R and consider the sequence $[x_n]$, where $x_n = fx_{n-1}$ for $n \geq 1$. For $m > n$

$$d(x_n, x_m) = d(f^n x_0, f^m x_0)$$

$$\leq d(f^n x_0, f^{n+1} x_0) + \ldots + d(f^{m-1} x_0, f^m x_0).$$

Since R is ε-chainable there is a chain, $x_0 = y_0, y_1, \ldots, y_r = fx_0$, such that $d(y_{k-1}, y_k) < \varepsilon$ for $1 \leq k \leq r$. Now $d(x_0, fx_0) \leq r\varepsilon$ and in general

$$d(f^k x_0, f^{k+1} x_0) \leq d(f^k y_0, f^k y_1) + \ldots + d(f^k y_{r-1}, f^k y_r)$$

$$\leq \alpha^k\, d(y_0, y_1) + \ldots + \alpha^k\, d(y_{r-1}, y_r)$$

$$\leq \alpha^k\, r\, \varepsilon.$$

Using the previous inequality with this result,

$$d(x_m, x_n) \leq d(f^n x_0, f^{n+1} x_0) + \ldots + d(f^{m-1} x_0, f^m x_0)$$

$$\leq \alpha^n\, r\, \varepsilon + \alpha^{n+1}\, r\, \varepsilon + \ldots + \alpha^{m-1}\, r\, \varepsilon$$

$$\leq r\, \varepsilon\, \alpha^n\, \frac{1}{1-\alpha}.$$

Since $0 \leq \alpha < 1$, $\lim \alpha^n = 0$. This implies that the sequence $[x_n]$ is a fundamental sequence and, by completeness of R, has a limit in R, say x.

The next step is to show that $fx = x$. Since f is a local contraction mapping on R, f is continuous. Thus

$$f(\lim x_n) = \lim fx_n, \text{ or } fx = \lim fx_n = \lim x_{n+1} = x,$$

and $fx = x$ as required.

To show uniqueness, suppose that there is another element in R, say y, such that $fy = y$ and $y \neq x$. By the properties of a metric, this implies that $d(x,y) \neq 0$. Now by chainability of R, there is a chain $x = z_0, z_1, \ldots, z_r = y$, such that $d(z_{i-1}, z_i) < \varepsilon$ for $1 \leq i \leq r$. Thus,

$$d(x,y) = d(fx,fy) = d(f^2x, f^2y) = \ldots = d(f^nx, f^ny) = \ldots$$

and

$$\begin{aligned} d(f^nx, f^ny) &\leq d(f^nz_0, f^nz_1) + \ldots + d(f^nz_{r-1}, f^nz_r) \\ &\leq \alpha^n\, d(z_0, z_1) + \ldots + \alpha^n\, d(z_{r-1}, z_r) \\ &\leq \alpha^n\, r\, \varepsilon\,. \end{aligned}$$

Thus for every $\varepsilon > 0$, $0 \leq d(x,y) \leq \varepsilon$. This implies that $d(x,y) = 0$, i.e., $x = y$. This is a contradiction and shows that the fixed point is indeed unique.

Remark. A result similar to Theorem 1-11 is proven by Edelstein in [1].

Definition 1-12. A metric space R is said to be connected if R cannot be covered by two non-empty, disjoint, open sets.

Lemma 1-13. If R is a connected metric space, then R is ε-chainable for every $\varepsilon > 0$.

Proof. The proof will be by contraposition. Let $\varepsilon > 0$ be given. Suppose that R is not ε-chainable. Then there exists elements a and b in R such that a cannot be ε-chained to b.

Let $A = [\, x: x \in R,\ x$ is not ε-chainable to $a\,]$ and let $B = R - A$, the complement of A in R. Clearly $A + B = R$ and $AB = 0$, the empty set.

Let x be an accumulation point of A. Then there exists an element y in A such that $d(x,y) < \varepsilon$. Since y is chainable to a, x is chainable to a. Thus x is in A, and A is closed.

Now let x be an accumulation point of B. Suppose that x is in A. Again there is an element y in B such that $d(x,y) < \varepsilon$. If x is in A, y is chainable to a and hence in A. This is impossible. Thus x must be in B, and B must be closed. It is now clear that A and B form a separation for R. Therefore R cannot be connected. This proves the lemma.

Theorem 1-14. If R is a complete, connected, metric space and $f: R \to R$ is an (ε, α)-uniformly locally contractive mapping, then there exists a unique element x in R such that $fx = x$.

Proof. This follows directly from Theorem 1-11 and Lemma 1-13.

Remark. The fixed point x mentioned in Theorem 1-14 is the limit of the sequence $[x_n]$, where x_1 is any element in R and $x_n = fx_{n-1}$ for $n > 1$.

CHAPTER II

EXPANDING MAPPINGS

<u>Definition 2-1</u>. Let $R_1 < R_2$ be subsets of a metric space R. A mapping $f:R_1$ onto R_2 is called an expanding mapping if there exists an $\alpha > 1$ such that $d(fx,fy) \geq \alpha\, d(x,y)$ for every pair of elements x and y in R.

<u>Theorem 2-2</u>. Let $R_1 < R_2$ be subsets of a metric space R. If $f:R_1$ onto R_2 is an expanding mapping then f is one to one, the inverse mapping f^{-1} exists and is a contraction mapping on R_2.

<u>Proof</u>. This is a direct result of the definition of an expanding mapping.

<u>Theorem 2-3</u>. Let $R_1 < R_2$ be metric spaces. Let $f:R_1$ onto R_2 be an expanding mapping. If R_2 is complete, there exists a unique element x in R_1 such that $fx = x$.

<u>Proof</u>. f^{-1} is a contraction mapping on R_2 and has one and only one fixed point.

<u>Remark</u>. The limit of the sequence $[y_n]$, where $y_1 \in R_1$ and $y_n = fy_{n-1}$ for $n > 1$, may not exist. This is seen by considering the mapping f, defined on the real line, such that $fx = 2x$ for $x \geq 0$.

<u>Definition 2-4</u>. Let $R_1 < R_2$ be metric spaces. A mapping $f:R_1$ onto R_2 is said to be a local expanding mapping if, for every element x in R_1, there is an $\varepsilon > 0$ and an $\alpha > 1$ such that

$$d(fy,fz) \geq \alpha\, d(y,z)$$

whenever y and z are in $N(x, \varepsilon)$.

Definition 2-5. Let $R_1 < R_2$ be metric spaces. A mapping $f: R_1$ onto R_2 is said to be (ε, α)-uniformly locally expanding if it is locally expanding and ε and α do not depend on the choice of x.

Remark. The analogue of Theorem 1-9 is not necessarily true for uniformly locally expanding mappings. To see this, consider the mapping $f:[-2,2]$ onto $[-4,4]$ where,

$$fx = 6 + 2x \qquad \text{for } x \text{ in } [-2,-1]$$

$$fx = 2x \qquad \text{for } x \text{ in } (-1,1)$$

$$fx = 6 - 2x \qquad \text{for } x \text{ in } [1,2].$$

Let $\varepsilon = \frac{1}{2}$ and $\alpha = 2$. Then f is a one to one, (ε, α)-uniformly locally expanding mapping but f is not a global expanding mapping. In addition f^{-1} is not a local contraction mapping.

This example shows that Corollary 6.1 of Edelstein [1] is not necessarily true unless f is continuous.

CHAPTER III

NON-EXPANDING MAPPINGS

Definition 3-1. A mapping f of a metric space R into itself is called a non-expanding mapping if, for every two elements x and y in R, $d(fx,fy) \leq d(x,y)$.

Theorem 3-2. Let R be a compact metric space and let f be the uniform limit of a sequence $[f_n]$, of contraction mappings, on R. Then f is a non-expanding mapping and has at least one fixed point.

Proof. Since f_n, for $n \geq 1$, is a contraction mapping on R, there is by completeness of R, for each n, a unique element x_n in R such that $f_n x_n = x_n$. Consider the sequence $[x_n]$. By compactness of R this sequence contains a convergent subsequence $[x_{n_i}]_{i=1}^{\infty}$ whose limit x is an element of R. f is continuous, being the uniform limit of a sequence of continuous functions, and

$$d(fx,f_{n_i}x_{n_i}) \leq d(fx,fx_{n_i}) + d(fx_{n_i},f_{n_i}x_{n_i}).$$

$$\lim d(fx,f_{n_i}x_{n_i}) \leq \lim d(fx,fx_{n_i}) + \lim d(fx_{n_i},f_{n_i}x_{n_i}),$$

if these limits exist. By continuity of f, $\lim d(fx,fx_{n_i}) = 0$.

By uniform convergence of $[f_n]$, $\lim d(fx_{n_i},f_{n_i}x_{n_i}) = 0$. Thus the limit of the left side of the last inequality exists and equals zero, i.e., $\lim d(fx,f_{n_i}x_{n_i}) = 0$. This implies that $fx = \lim f_{n_i}x_{n_i} = x$.

To prove that f is a non-expanding mapping, let x and y be any two points in R. Then

$$d(fx,fy) \leq d(fx,f_nx) + d(f_nx,f_ny) + d(f_ny,fy)$$

$$\leq d(fx,f_nx) + \alpha_n\, d(x,y) + d(f_ny,fy)$$

where α_n denotes the contraction constant for the mapping f_n. Taking the lim sup of each side of this inequality, $d(fx,fy) \leq d(x,y)$. Thus f is a non-expanding mapping on R.

Remark. A non-expanding mapping on a compact metric space need not have a fixed point. To see this let

$$R = [\ x: x \in E_2 \text{ and } d(x,0) = 1\ ]$$

where d denotes the Euclidean distance. By rotating each point in R through $\frac{\pi}{2}$ radians, it is clear that no point remains fixed.

Corollary 3-3. Let $K = [\ x: x \in R,\ ||x|| \leq 1\ ]$ be a subset of a finite dimensional, complete, normed linear space R. If f is a non-expanding mapping of K, there exists at least one element, x in K, such that fx = x.

Proof. Consider, for each integer $n \geq 1$, the mapping $f_n = (1 - \frac{1}{n})f$. Each $f_n: K \to K$, for $n \geq 1$, is a contraction mapping. For each x in K,

$$||(1 - \tfrac{1}{n})fx - fx|| = ||\tfrac{1}{n}fx|| = \tfrac{1}{n}||fx|| \leq \tfrac{1}{n}.$$

Therefore, the sequence $[f_n]$ converges uniformly to f. Since K is compact, Theorem 3-2 applies and the proof is complete.

Remark. Corollary 3-3 is proved directly by Bers in [2].

Remark. Non-expanding mappings will be discussed further in Chapter IV.

CHAPTER IV

ISOMETRIES

Definition 4-1. A metric space R is said to be totally bounded if, for every $\epsilon > 0$, R can be covered by a finite number of sets R_i such that the diameter of each R_i is smaller than ϵ.

Definition 4-2. A metric space R is said to be compact if every open covering of R contains a finite subcovering.

Remark. It is well known that a continuous mapping takes a compact metric space into a compact space. See for example Hall and Spencer [3].

Remark. If a metric space R is compact, then R is totally bounded.

Lemma 4-3. If R is a totally bounded metric space then every sequence of elements of R contains a fundamental subsequence.

Proof. Consider the sequence $[x_n]$ where $x_n \in R$, for $n \geq 1$. Let $\epsilon_k = \frac{1}{k}$ for $k = 1, 2, \ldots$. By total boundedness of R there is, for each ϵ_k, a finite number of sets R_i^k of diameter less than ϵ_k, which cover R. For $k = 1$ at least one of these sets, say R_1^1, contains an infinite subsequence $[x_n^1]$ of $[x_n]$. R_1^1 must also be a totally bounded set, being a subset of R. For $k = 2$, R_1^1 can be covered by a finite number of sets with diameter less than $\frac{1}{2}$. One of these sets, say R_1^2, must contain an infinite subsequence $[x_n^2]$ of $[x_n^1]$. Continue this process indefinitely and form the array

$$\begin{array}{ccccc} x_1^1 & x_2^1 & \ldots & x_n^1 & \ldots \\ x_1^2 & x_2^2 & \ldots & x_n^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^n & x_2^n & \ldots & x_n^n & \ldots \end{array}$$

where the elements in the kth. row are those obtained at the kth. step. Consider the subsequence formed from the diagonal elements of the array, i.e., $[x_n^n]$. Clearly this is a fundamental sequence since for every $\varepsilon > 0$, an integer $N > 0$ can be found such that $d(x_n^n, x_m^m) < \varepsilon$ whenever $m, n > N$.

Theorem 4-4. Let R be a totally bounded metric space. Let $f: R \to R$ be a mapping such that $d(fx, fy) \geq d(x, y)$ for every pair of elements x and y in R. Then $d(fx, fy) = d(x, y)$. If R is compact, $f(R) = R$.

Proof. Let a and b be any two points in R and let $a_0 = a$, $b_0 = b$. Consider the sequences $[a_n]$ and $[b_n]$ where $a_n = fa_{n-1}$ and $b_n = fb_{n-1}$ for $n \geq 1$. Since R is totally bounded there is a subsequence $[a_{n_i}]$ of $[a_n]$, which is a fundamental sequence.

Consider the sequence $[b_{n_i}]$. This sequence also contains a fundamental subsequence say $[b_{n_{i_j}}]$. The sequence $[a_{n_{i_j}}]$, a subsequence of $[a_{n_i}]$, must be a fundamental sequence. Thus, for every $\varepsilon > 0$, there exists an integer i and an integer k such that,

$$d(a_i, a_{i+k}) < \frac{\varepsilon}{2} \text{ and } d(b_i, b_{i+k}) < \frac{\varepsilon}{2}.$$

$$d(a_0,a_k) \leq d(a_1,a_{k+1}) \leq \ldots \leq d(a_i,a_{i+k}) < \frac{\varepsilon}{2}$$

and similarly $d(b_0,b_k) < \frac{\varepsilon}{2}$. Therefore

$$d(a_1,b_1) \leq d(a_k,b_k) \leq d(a_k,a_0) + d(a_0,b_0) + d(b_0,b_k)$$

$$< \frac{\varepsilon}{2} + d(a_0,b_0) + \frac{\varepsilon}{2}$$

$$= \varepsilon + d(a_0,b_0).$$

Since $\varepsilon > 0$ is arbitrary, $d(a_1,b_1) \leq d(a_0,b_0)$. By hypothesis

$$d(a_0,b_0) \leq d(a_1,b_1).$$

Therefore,

$$d(a_1,b_1) \leq d(a_0,b_0) \leq d(a_1,b_1) \text{ and } d(a_0,b_0) = d(a_1,b_1).$$

This proves the first part of the theorem.

Let R be compact. Suppose that there exists a point x_0 in R such that x_0 is not in $f(R)$. Since f is continuous, $f(R)$ is compact. Thus $d(x_0,f(R)) = \varepsilon > 0$. Consider the sequence $[x_n]$ where $x_1 = fx_0$ and $x_n = fx_{n-1}$ for $n \geq 1$. This sequence is contained in $f(R)$ which is compact. Therefore, $[x_n]$ has a convergent subsequence. For all integers m and n such that $0 \leq n < m$,

$$d(x_n,x_m) = d(x_{n-1},x_{m-1}) = \ldots = d(x_0,x_{m-n})$$

$$= d(x_0,fy) \geq d(x_0,f(R)) = \varepsilon$$

for some y in $f(R)$. Therefore, $d(x_n,x_m) \geq \varepsilon > 0$ and $[x_n]$ cannot contain a convergent subsequence. This contradiction shows that $f(R) = R$.

Remark. The first part of Theorem 4-4 is given by Freudenthal and Hurewicz in [4].

Theorem 4-5. Let R be a totally bounded metric space. If the mapping $f:R$ onto R is such that $d(fx,fy) \leq d(x,y)$ for every pair of elements x and y in R, then $d(fx,fy) = d(x,y)$ for all elements x and y in R.

Proof. It will be shown that f is a one to one mapping on R. Then f^{-1} exists, $f^{-1}(R) = R$, and $d(f^{-1}x,f^{-1}y) \geq d(x,y)$ for all elements x and y in R. Thus by Theorem 4-4, $d(f^{-1}x,f^{-1}y) = d(x,y)$ and the proof will be complete.

Suppose that there are two points, x and y in R, such that $fx = fy$. Since R is totally bounded, a finite open covering, consisting of open spheres of radius ε, can be found for each $\varepsilon > 0$. Let $[N(c_k,\varepsilon)]$ be such a covering where c_k, $k = 1,2,\ldots,m$ denotes the centers of the spheres. Since

$$\sum_{k=1}^{n} N(fc_k,\varepsilon) > \sum_{k=1}^{n} f(N(c_k,\varepsilon)) = f(\sum_{k=1}^{n} (N(c_k,\varepsilon))) > f(R) = R,$$

$[N(fc_k,\varepsilon)]$ is an open covering of R. Now let $d(x,y) = k > 0$ and choose $\varepsilon = \frac{k}{5}$. Let n be the smallest number of open spheres, with radius ε that covers R. For each such covering let l denote the sum of the distances between all the centers of all the covering spheres, i.e.,

$$l = \sum_{i<j}^{n} d(c_i,c_j) \geq k - 2\varepsilon = k - \frac{2k}{5} = \frac{3k}{5} > 0.$$

Let l_m denote the infimum of all values of l taken over all coverings of this type. Clearly $l_m \geq \frac{3k}{5}$. Let $N(c_k^*, \varepsilon)$ for $k = 1,2,\ldots,n$ denote a covering of R such that

$$\sum_{i<j}^{n} d(c_i^*, c_j^*) < l_m + \varepsilon .$$

Without loss of generality assume that $x \in N(c_1^*, \varepsilon)$ and that $y \in N(c_2^*, \varepsilon)$. Now if $fx = fy$,

$$\sum_{i<j}^{n} d(fc_i^*, fc_j^*) = \sum_{\substack{i<j \\ (i\neq 1,\, j\neq 2)}}^{n} d(fc_i^*, fc_j^*) + d(fc_1^*, fc_2^*)$$

$$< \sum_{\substack{i<j \\ (i\neq 1,\, j\neq 2)}}^{n} d(c_i^*, c_j^*) + 2\varepsilon$$

$$< l_m + \varepsilon - 3\varepsilon + 2\varepsilon = l_m .$$

But this is clearly a contradiction to the fact that l_m is the infimum of all such coverings. This implies that $fx \neq fy$ for any two points x and y in R. Our previous remarks are applicable and the proof of the theorem is complete.

Remark. Theorem 4-5 is stated, without proof, by Freudenthal and Hurewicz in [4].

Theorem 4-6. Let R be a totally bounded metric space. Let f:R onto R be a mapping such that $d(fx,fy) \leq d(x,y)$ whenever x and y are in R and $d(x,y) < \varepsilon$ for some fixed $\varepsilon > 0$. Then $d(fx,fy) = d(x,y)$ whenever x and y are in R and $d(x,y) < \varepsilon$.

Proof. A proof of Theorem 4-6 is given by Edrei in [5].

Remark. The first part of Theorem 4-5 could be taken as a corollary to Theorem 4-6.

Example 4-7. Let f:[-2,2] onto [-2,2] be the mapping:

$$fx = 3 + x \qquad \text{for } x \text{ in } [-2,-1]$$

$$fx = x \qquad \text{for } x \text{ in } (-1,1)$$

$$fx = 3 - x \qquad \text{for } x \text{ in } [1,2].$$

Let $\varepsilon = \frac{1}{2}$ and d represent the Euclidean one dimensional distance. If x and y are in [-2,2] and $d(x,y) < \varepsilon$, then $d(fx,fy) \geq d(x,y)$. It is not true however that $d(fx,fy) \geq d(x,y)$ for every x and y in R. Nor is it true that there is a neighborhood $N(x,\varepsilon_x)$ about each point x in R such that y,z in $N(x,\varepsilon_x)$ implies $d(x,z) = d(fy,fz)$.

Example 4-8. Let R = [0,1,3] be a metric space under the Euclidean distance. Let f:R onto R be the mapping: f(0) = 1, f(1) = 3, f(3) = 0. Let $\varepsilon = \frac{3}{2}$. Then $d(x,y) < \varepsilon$ implies $d(fx,fy) \geq d(x,y)$ for all x,y in R, but $d(x,y) < \varepsilon$ does not imply that $d(fx,fy) = d(x,y)$. Since f is continuous this clearly contradicts Corollary 3.8 of Brown and Comfort [6].

Remark. Example 4.7 and 4.8 suggest the following conjecture which the author has been anable to prove.

Conjecture 4-9. Let R be a compact metric space. Let $f:R \to R$ be a continuous mapping such that $d(fx,fy) \geq d(x,y)$ whenever $d(x,y) > \varepsilon > 0$. Then there exists a $\beta > 0$ such that $d(fx,fy) = d(x,y)$ whenever $d(x,y) < \beta$.

Remark. Let R be a compact metric space. Williams in [7] displayed a mapping f:R onto R such that $d(fx,fy) \leq d(x,y)$ whenever y in $N(x,\varepsilon_x)$ but $d(fz,fx) \neq d(x,z)$ for every z in $N(x,\beta_x)$ for any $\beta_x > 0$. This suggests the following theorem.

Theorem 4-10. Let R be a compact metric space and f a mapping of R onto R. If for every x in R there exists an $\varepsilon > 0$ such that $d(fy,fz) \leq d(y,z)$ whenever y and z are in $N(x,\varepsilon)$, then there exists a $\beta > 0$ such that $d(fy,fz) = d(y,z)$ whenever y and z are in R and $d(y,z) < \beta$.

Proof. It is first shown that there exists a $\beta > 0$ such that $d(fy,fz) \leq d(y,z)$ whenever $d(y,z) < \beta$. Let $\beta_n = \frac{1}{n}$, for all integers $n \geq 1$. Suppose no such β exists. Then for each β_n, $n \geq 1$ there exists an x_n and a y_n which are elements of R, such that $d(x_n,y_n) < \frac{1}{n}$ and $d(fx_n,fy_n) > d(x_n,y_n)$. Consider the sequences $[x_n]$ and $[y_n]$. Because R is compact we can construct (as in the proof of Theorem 4-4) two convergent subsequences $[x_{n_i}]$ and $[y_{n_i}]$. Both subsequences have a common limit. Call this limit x. Then $\lim x_{n_i} = \lim y_{n_i} = x$ and for each $i \geq 1$, $d(fx_{n_i},fy_{n_i}) > d(x_{n_i},y_{n_i})$. Because of the convergence of $[x_{n_i}]$ and $[y_{n_i}]$, for some m_i, x_{m_i} and y_{m_i} are both contained in $N(x,\varepsilon_x)$where $N(x,\varepsilon_x)$ is chosen as in the hypothesis. This implies that $d(fx_{m_i},fy_{m_i}) \leq d(x_{m_i},y_{m_i})$ and contradicts the fact that $d(fx_{m_i},fy_{m_i}) > d(x_{m_i},y_{m_i})$. Hence $\beta > 0$ exists.

The existence of a $\beta > 0$, for which $d(fx,fy) \leq d(x,y)$ whenever $d(x,y) < \beta$, proves, by Theorem 4-6, the theorem.

CHAPTER V

NORMS

Definition 5-1. A linear space R over the complex field F is said to be normed if to each element x in R there is made to correspond a nonnegative number $||x||$ which is called the norm of x and such that:

$||x|| = 0$ if, and only if, $x = 0$

$||cx|| = |c|\ ||x||$ for each c in F

$||x + y|| \leq ||x|| + ||y||$ for all x and y in R

Remark. Hereafter, it will be assumed that a linear space R is taken over the complex field unless otherwise stated.

Definition 5-3. Let R be a normed linear space and let $A:R \rightarrow R$ be a linear function. A is said to be bounded if there exists a constant M such that $||Ax|| \leq M\ ||x||$ for all x in R.

Definition 5-4. Let R be a normed linear space and let $A:R \rightarrow R$ be a bounded linear function. The norm of A, written $||A||$, is the greatest lower bound of all numbers M satisfying $||Ax|| \leq M\ ||x||$ for all elements x in R.

Theorem 5-5. If $A:R \rightarrow R$ is a bounded linear function and R a normed linear space, $||A|| = \sup[\ ||Ax|| : ||x|| = 1\ ]$.

Proof. See Kolmogorov and Fomin [8].

Remark. $\sup[\ ||Ax|| : ||x|| = 1\ ] = \sup[\ ||Ax||/||x|| : ||x|| \neq 0]$

Theorem 5-6. Let R be a normed linear space. If $A: R \to R$ and $B: R \to R$ are linear functions:

$$\|A\| \geq 0,\ \|A\| = 0 \text{ if, and only if, } A = 0 \tag{5.1}$$

$$\|AB\| \leq \|A\|\ \|B\| \tag{5.2}$$

$$\|cA\| = |c|\ \|a\| \text{ for all } c \text{ in } F \tag{5.3}$$

$$\|A + B\| \leq \|A\| + \|B\| \tag{5.4}$$

$$\|Ax\| \leq \|A\|\ \|x\| \text{ for all } x \text{ in } R \tag{5.5}$$

Proof. See Kolmogorov and Fomin [8].

Remark. (5.1) through (5.4) are frequently taken as defining properties for a matrix norm.

Theorem 5-7. Let R_n be the linear space of all real n-tuples taken over the real field. Let $A: R_n \to R_n$ be a linear function. Let (a_{ij}) be the matrix representation of A and let $x = (x_1, \ldots, x_n)^T$ be any vector in R_n. Let :

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{\frac{1}{2}} \tag{5.6}$$

$$\|A\|_2 = \sqrt{\lambda} \tag{5.7}$$

where λ is the largest eigenvalue of the matrix AA^T.

$$\|x\|_\infty = \max_i |x_i| \tag{5.8}$$

$$\|A\|_\infty = \max_i \sum_{k=1}^{n} |a_{ik}| \tag{5.9}$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \tag{5.10}$$

$$\|A\|_1 = \max_k \sum_{i=1}^{n} |a_{ik}| \tag{5.11}$$

Then $(R_n, \|\ \ \|_j)$ for $j = 1, 2, \infty$ is a normed linear space. If $A: (R_n, \|\ \ \|_j) \to (R_n, \|\ \ \|_j)$ is a linear function, then $\|A\|_j$ for $j = 1, 2, \infty$ is the linear function norm on the respective space.

<u>Proof.</u> See Faddeeva [9].

<u>Definition 5-8.</u> The normed linear space $(R_n, \|\ \ \|_j)$ for $j = 1, 2, \infty$ will be written l_1^n, l_2^n (or E_n), and l_∞^n respectively.

CHAPTER VI

PROBABILITY

Definition 6-1. A real n by n matrix A is called a stochastic matrix if $0 \leq a_{ij} \leq 1$ for each element a_{ij} of A and if $\sum_{j=1}^{n} a_{ij} = 1$ for $1 \leq i \leq n$.

Definition 6-2. An n by n stochastic matrix A is said to be a regular transition matrix if each element of A^n is greater than zero for some integer $n \geq 1$. If each element of a matrix B is greater than zero, we will write $B > 0$.

Remark. If A is a stochastic matrix then A^n is a stochastic matrix for all integers $n \geq 1$.

Definition 6-3. A vector $x = (x_1, \ldots, x_n)$ is said to be a probability vector if $0 \leq x_i \leq 1$ for $1 \leq i \leq n$ and if

$$\sum_{i=1}^{n} x_i = 1.$$

Theorem 6-4. Let A be an n by n regular transition matrix and R the set of all nth. order probability vectors. Then $A(R) < R$ and under the l_1^n norm, A^n is a contraction mapping on R for some integer $n \geq 1$.

Proof. By the definition of matrix multiplication,

$$(x_1, \ldots, x_n)\ (a_{ij}) = \left(\sum_{k=1}^{n} x_k a_{k1}, \ldots, \sum_{k=1}^{n} x_k a_{kn} \right).$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} x_k a_{kj} = \sum_{k=1}^{n} x_k \sum_{j=1}^{n} a_{kj} = 1. \text{ Thus } A(R) < R.$$

Since A^n is a stochastic matrix for $n \geq 1$, $A^n(R) < R$. Also, because A is a regular transition matrix, $A^n > 0$ for some integer $n \geq 1$. Let n be the smallest integer such that $A^n > 0$ and let $A^n = B = (b_{ij})$.

$||xB - yB||_1 = \sum_{j=1}^{n} |\sum_{k=1}^{n} b_{kj}(x_k - y_k)|$ for any elements x and y in R.

Let $S = [j: \sum_{k=1}^{n} b_{kj}(x_k - y_k) \geq 0]$ and let

$T = [j: \sum_{k=1}^{n} b_{kj}(x_k - y_k) < 0]$. Then

$$||xB - yB||_1 = \sum_{j\epsilon S} (\sum_{k=1}^{n} b_{kj}(x_k - y_k)) - \sum_{j\epsilon T} (\sum_{k=1}^{n} b_{kj}(x_k - y_k))$$

$$= \sum_{k=1}^{n} (\sum_{j\epsilon S} b_{kj}(x_k - y_k)) - \sum_{k=1}^{n} (\sum_{j\epsilon T} b_{kj}(x_k - y_k))$$

$$= \sum_{k=1}^{n} (x_k - y_k)[\sum_{j\epsilon S} b_{kj} - (1 - \sum_{j\epsilon S} b_{kj})]$$

$$= \sum_{k=1}^{n} (x_k - y_k)[2\sum_{j\epsilon S} b_{kj} - 1].$$

Case 1. S is the empty set.

$$||xB - yB||_1 = -\sum_{j=1}^{n} (\sum_{k=1}^{n} b_{kj}(x_k - y_k)) = -\sum_{k=1}^{n} (\sum_{j=1}^{n} b_{kj}(x_k - y_k))$$

$$= -\sum_{k=1}^{n} (x_k - y_k) = 0$$

and $||xB - yB||_1 \leq \alpha \, ||x - y||_1$ for all positive numbers α.

Case 2. T is the empty set.

$$||xB - yB||_1 = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} b_{kj}(x_k - y_k) \right) = 0$$

and $||xB - yB||_1 \leq \alpha \, ||x - y||_1$ for all positive numbers α.

Case 3. There exists at least one j in S and at least on j in T.

$$||xB - yB||_1 = \sum_{k=1}^{n} (x_k - y_k)[2 \sum_{j \in S} b_{kj} - 1].$$

$\min_{i,k} b_{ik} \leq \sum_{j \in S} b_{kj} \leq \sum_{\substack{j=1 \\ j \neq r}}^{n} b_{kj}$ where b_{kr} is the smallest element in the kth. row. Since $b_{ik} > 0$ for all i,k we have by the triangle inequality,

$$||xB - yB||_1 \leq \sum_{k=1}^{n} |x_k - y_k| \; |2 \sum_{j \in S} b_{kj} - 1| \leq \alpha \sum_{k=1}^{n} |x_k - y_k|$$

where $\alpha = \max \left(|2 \min_{i,j} b_{ij} - 1|, \; 2[\max_k \sum_{\substack{j=1 \\ j \neq r}}^{n} b_{kj}] - 1 \right)$ and $0 \leq \alpha \leq 1$.

Corollary 6-5. Let A be a regular n by n transition matrix and R the set of all nth. order probability vectors. Then there exists a unique element x in R such that $xA = x$. x is the limit of the sequence $[x_n]$, where x_1 is any element of R and $x_n = x_{n-1}A$ for $n \geq 1$.

Proof. This result follows immediately from Theorem 6-4 and Corollary 1-6.

Remark. It is easy to show that the results of Theorem 6-4 are not necessarily true under the l_2^3 or l_∞^4 norm.

Theorem 6-6. Let $A > 0$ be a 2 by 2 stochastic matrix and R the set of all second order probability vectors. Then A is a contraction mapping on R under the l_1^n, l_2^n, and l_∞^n norm.

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any two elements in R and let $z = x - y = (z_1, z_2)$. Let

$$A = \begin{bmatrix} a_{11} & (1 - a_{11}) \\ a_{21} & (1 - a_{21}) \end{bmatrix}$$

Noting that $z_2 = - z_1$, $zA = ((a_{11} - a_{21})z_1, (a_{21} - a_{11})z_1)$.

Case 1. l_∞^n norm. $||x - y||_\infty = |z_1|$ and

$$||xA - yA||_\infty = \max\ (|a_{11} - a_{21}|\ |z_1|, |a_{21} - a_{11}|\ |z_1|).$$

Therefore $||xA - yA||_\infty \leq \alpha\ ||x - y||_\infty$ where $0 \leq \alpha < 1$.

Case 2. l_2 norm. $||x - y||_2 = |z_1|\ (2)^{\frac{1}{2}}$ and

$$||xA - yA||_2 = |z_1|\ ((a_{11} - a_{21})^2 + (a_{21} - a_{11})^2)^{\frac{1}{2}}$$

$$\leq \alpha\ ||x - y||_2 \text{ where } 0 \leq \alpha < 1.$$

Case 3. l_1 norm. $||x - y||_1 = 2|z_1|$ and

$$||xA - yA||_1 = |z_1|\ [\ |a_{11} - a_{21}| + |a_{21} - a_{11}|\ ]$$

$$\leq \alpha\ ||x - y||_1 \text{ where } 0 \leq \alpha < 1.$$

CHAPTER VII

MATRIX ANALYSIS

Theorem 7-1. Let R be a finite dimensional, complete normed linear space. A linear function $A: R \to R$ is a contraction mapping if and only if $||A|| < 1$.

Proof. The sufficiency is clear since $||Ax|| \leq ||A||\ ||x||$ for every element x in R.

To prove necessity it suffices to show that there exists a non-zero vector y in R such that $||Ay|| = ||A||\ ||y||$. $||A|| = \sup [\ ||Ax|| : ||x|| = 1\ ]$ from Theorem 5-5. Hence there is a sequence $[x_n]$ such that $\lim ||Ax_n|| = ||A||$ and $||x_n|| = 1$ for $n \geq 1$. Since $S = [\ x: ||x|| = 1\ ]$ is a compact set there is a subsequence of $[x_n]$, say $[y_n]$, such that $\lim y_n$ exists. Call this limit y. $\lim ||Ay_n|| = ||A||$ and $||y_n|| = 1$ for $n \geq 1$. By the continuity of a norm, $||\lim Ay_n|| = ||A||$ and by continuity of A (since A is bounded), $||A \lim y_n|| = ||Ay|| = ||A||$. Since $||y|| = 1$, $||Ay|| = ||A||\ ||y||$ as required.

Theorem 7-2. Let $A: R \to R$ be a linear function and R a complete, normed linear space. If A is a contraction mapping, then $Ax = x$ for some x in R, if and only if x is the zero vector.

Proof. Since $A: R \to R$ is a contraction mapping, there exists a unique element x in R such that $Ax = x$. $A0 = 0$ because A is a linear function.

Theorem 7-3. Let $A: R \to R$ be a bounded linear function and R a normed linear space. Then all eigenvalues of A are in absolute value less than or equal to the norm of A.

Proof. If λ is an eigenvalue of A then there exists a non-zero vector x in R such that $Ax = \lambda x$. Thus $||Ax|| = ||\lambda x|| = |\lambda|\ ||x||$. But $||Ax|| \leq ||A||\ ||x||$ holds for all elements x in R. So $|\lambda|\ ||x|| \leq ||A||\ ||x||$ and $|\lambda| \leq ||A||$.

Corollary 7-4. Let $A: R \to R$ be a linear function and R a finite dimensional, complete, normed linear space. If any eigenvalue of A is greater than or equal to one, A cannot be a contraction mapping on R.

Proof. This is a direct result of Theorem 7-3 and Theorem 7-1.

Example 7-5. Let $A = (a_{ij}) > 0$ be any 3 by 3 stochastic matrix (see Chapter VI). Let $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, and $x_3 = (0,0,1)$ be a basis for E_3. Under the l_2^3 norm, $||x_i - x_j||_2^2 = 2$ if $i \neq j$.

$$||x_iA - x_jA||_2^2 = (a_{i1} - a_{j1})^2 + (a_{i2} - a_{j2})^2 + (a_{i3} - a_{j3})^2$$

$$= a_{i1}^2 + a_{i2}^2 + a_{i3}^2 + a_{j1}^2 + a_{j2}^2 + a_{j3}^2$$

$$- 2[\ a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3} + a_{i3}a_{j3}\ ]$$

$$< 2 \qquad \text{for } i,j = 1,2,3.$$

Thus $||x_iA - x_jA||_2 \leq \alpha\ ||x_i - x_j||_2$ for $i,j = 1,2,3$ where $0 \leq \alpha < 1$.

Remark. In Example 7-5, $A:E_3 \to E_3$ is a linear function and contracts the basis vectors x_1, x_2, x_3. It is clear by the remarks following Corollary 6-5 that A need not be a contraction mapping on E_3. Moreover, the remark shows that $A:H \to H$ is not necessarily a contraction mapping on the convex hull H of x_1, x_2, x_3.

Theorem 7-6. Let $A:R_n \to R_n$ be a real n by n matrix. If $||A + I|| < 1$, the matrix equation $Ax = b$ has a unique solution in R_n. The solution x is the limit of the sequence $[x_n]$ where x_0 is any element in R and $x_n = (A + I)x_{n-1} - b$ for $n \geq 1$.

Proof. Let $B:R_n \to R_n$ be the function, defined for each x in R_n, so that $Bx = (A + I)x - b$. For any two vectors x and y in R_n, $Bx - By = (A + I)\ (x - y)$ and

$$||Bx - By|| = ||(A + I)(x - y)|| \leq ||A + I||\ ||x - y||.$$

If $||A + I|| < 1$, there exists a unique vector x in R such that $Bx = x$, i.e., $Ax = b$. Theorem 1-4 guarantees that x will be given as the limit of the sequence $[x_n]$ where x_0 is any element in R_n and $x_n = (A + I)x_{n-1} - b$ for $n \geq 1$.

Theorem 7-7. A sufficient condition that a real n by n matrix A be non-singular is that $||A + I|| < 1$.

Proof. By Theorem 7-6, there is for each b in R_n, a unique solution to the equation $Ax = b$ when $||A + I|| < 1$. This implies that A inverse exists and hence that A is non-singular.

Theorem 7-8. Let $A:R \to R$ be a linear function and R a complete, finite dimensional, normed linear space. If $||A|| = 1$, the sequence $[S_n(x)]$, where

$$S_n(x) = \frac{Ax + \ldots + A^n x}{n} \qquad \text{for } n \geq 1,$$

has a convergent subsequence $[S_{n_i}(x)]$ for all x in R. Let

$S(x) = \lim S_{n_i}(x)$. Then $A\,S(x) = S(x)$.

Proof. For each integer $n \geq 1$,

$$\begin{aligned}||S_n(x)|| &= \left|\left|\frac{Ax + \ldots + A^n x}{n}\right|\right| \\ &\leq \frac{1}{n}(||A||\ ||x|| + \ldots + ||A||^n\ ||x||) \\ &= ||x||.\end{aligned}$$

Hence $[S_n(x)]$ is a bounded sequence contained in R. This implies the existence of a convergent subsequence $[S_{n_i}(x)]$. Let

$$S(x) = \lim S_{n_i}(x).$$

Since A is bounded, A is continuous and

$$A \lim S_{n_i}(x) = AS(x)$$

$$\lim AS_{n_i}(x) = AS(x).$$

$$\begin{aligned}\lim AS_{n_i}(x) &= \lim A\left(\frac{Ax + \ldots + A^{n_i}x}{n_i}\right) \\ &= \lim \frac{A^2x + \ldots + A^{n_i+1}x}{n_i} \\ &= \lim\left(\frac{Ax + A^2x + \ldots + A^{n_i}x}{n_i} + \frac{A^{n_i+1}x - Ax}{n_i}\right)\end{aligned}$$

$$= \lim \left(\frac{Ax + A^2x + \ldots + A^{n_i}x}{n_i}\right) + \lim \left(\frac{A^{n_i+1}x - Ax}{n_i}\right)$$

$$= S(x).$$

Remark. If for any positive integer k, $||A^k|| < ||A||^k = 1$, then by Theorem 7-1, A^k is a contraction mapping and $S(x) = 0$.

Definition 7-9. Let $A: R \to R$ be a bounded linear function on a normed linear space R. Then there exists an $M \geq 0$ such that $||Ax|| \geq M||x||$ for all x in R. Let [[A]] designate the supremum of all M such that $||Ax|| \geq M||x||$ for all x in R.

Remark. Let $[[A]]_j$ for $j = 1,2,\infty$ correspond to the normed linear space l_1^n, l_2^n, l_∞^n respectively.

Theorem 7-10. Let $A: R \to R$ be a bounded linear function and R a normed linear space. Then $[[A]] = \inf [\ ||Ax|| : ||x|| = 1\ ]$.

Proof. Let $a = \inf [\ ||Ax|| : ||x|| = 1\ ]$

$$= \inf [\ ||Ax||/||x|| : ||x|| \neq 0\ ].$$

$a = \inf [\ ||Ax||/||x|| : ||x|| \neq 0\ ]$ implies $||Ax|| \geq a||x||$

for every x in R such that $||x|| \neq 0$. Thus $a = [[a]]$.

Definition 7-11. Let $A: R \to R$ be a linear function and R a finite dimensional normed linear space. Denote by $|\lambda|_m$ the smallest in absolute value of all eigenvalues of A.

Theorem 7-12. Let $A: R \to R$ be a linear function and R a finite dimensional normed linear space. Then $0 \leq [[A]] \leq |\lambda|_m$.

Proof. For each eigenvalue λ of A there is at least one non-zero vector x such that $ax = \lambda x$. Thus $||Ax|| = |\lambda|_m||x||$

for some x in R. This implies that $[[A]] \leq |\lambda|_m$.

Theorem 7-13. If $A: R_n \to R_n$ is an n by n diagonal matrix then $[[A]] = |\lambda|_m$ under the l_1^n, l_2^n, and l_∞^n norms.

Proof. Let $A = (\delta_{ij}\lambda_i)$.

Case 1. l_1^n norm.

$$||Ax||_1 = \sum_{k=1}^{n} |\lambda_k x_k| \geq |\lambda|_m \sum_{k=1}^{n} |x_k| = |\lambda|_m ||x||_1.$$ Therefore,

$||Ax||_1 \geq |\lambda|_m ||x||_1$ and $|\lambda|_m \leq [[A]]$. By Theorem 7-12 $[[A]]_1 \leq |\lambda|_m$.

Thus $|\lambda|_m = [[A]]$.

Case 2. l_2^n norm.

$$||Ax||_2^2 = \sum_{k=1}^{n} \lambda_k^2 x_k^2 \geq |\lambda|_m^2 \sum_{k=1}^{n} x_k^2 = (|\lambda|_m ||x||_2)^2.$$

Therefore, $||Ax||_2 \geq |\lambda|_m ||x||_2$. The remainder of the proof is similar to case 1.

Case 3. l_∞^n norm.

$$\max_k |\lambda_k x_k| \geq \max_k |\lambda|_m |x_k| = |\lambda|_m \max_k |x_k|.$$

Therefore, $||Ax||_\infty \geq |\lambda|_m ||x||_\infty$. The remainder of the proof is similar to case 1.

Remark. Let $x = (1,2,1)^T$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 2 & 0 \\ -1 & -1 & 3 \end{bmatrix}$$

The matrix $A: R_n \to R_n$ is such that $[[A]]_j \neq |\lambda|_m$ for $j = 1, 2, \infty$. This is seen by considering Ax.

Theorem 7-14. Let $A: R \to R$ be a non-singular, linear function and R a finite dimensional, complete, normed linear space. Then

$$[[A]] = \frac{1}{||A^{-1}||}.$$

Proof. Since A is non-singular, A^{-1} exists and $||A^{-1}x|| \leq ||A^{-1}||\ ||x||$ for each x in R. For any x in R there is an element y in R such that $x = Ay$. Therefore $||y|| \leq ||A^{-1}||\ ||Ay||$. Since $||A^{-1}|| \neq 0$,

$$||Ay|| \geq \frac{1}{||A^{-1}||}\ ||y|| \text{ for all } y \text{ in } R.$$ This means that

$$[[A]] \geq \frac{1}{||A^{-1}||}.$$

It was shown in the proof of Theorem 7-1 that $||A^{-1}x_0|| = ||A^{-1}||\ ||x_0||$ for some non-zero element x_0 in R. Let $y_0 = A^{-1}x_0$. Then

$$||Ay_0|| = \frac{1}{||A^{-1}||}||y_0|| \text{ and } [[A]] = \frac{1}{||A^{-1}||}.$$

Corollary 7-15. Let A be a non-singular, linear function on a finite dimensional, complete, normed linear space R. Then

$$||A^{-1}||^{-1} = \inf [\ ||Ax|| : ||x|| = 1\ ].$$

Proof. This is a direct result of Theorem 7-10 and Theorem 7-14.

Lemma 7-16. Let $A: R_n \to R_n$ be a real, non-singular, n by n matrix. Then

$$||A^{-1}||_2 = \frac{1}{\sqrt{(\lambda_m^*)}} \quad ,$$

where λ_m^* represents the smallest eigenvalue of AA^T.

Proof. AA^T is a positive definite, real, symmetric matrix. There exists a non-singular matrix P such that $P^{-1}AA^TP = D$, a diagonal matrix. D has the same eigenvalues as AA^T.

Consider now $P^{-1}(AA^T)^{-1}P$. Clearly $P^{-1}(AA^T)^{-1}P = (P^{-1}AA^TP)^{-1}$. Hence $P^{-1}(AA^T)^{-1}P$ has eigenvalues which are the reciprocals of the eigenvalues $(AA^T)^{-1}$. This means that the eigenvalues of AA^T are reciprocals of those of $(AA^T)^{-1}$. Consider $(A^{-1})(A^{-1})^T$. $(A^{-1})(A^{-1})^T = A^{-1}(A^T)^{-1} = (A^TA)^{-1}$. Since A is non-singular $A^{-1}(AA^T)A = A^TA$. This implies that the eigenvalues of A^TA are equal to the eigenvalues of AA^T. Hence the eigenvalues of $(A^TA)^{-1}$ are equal to the eigenvalues of $(AA^T)^{-1}$. Thus the eigenvalues of $(A^{-1})(A^{-1})^T$ are reciprocals of the eigenvalues of AA^T. Because of 5.7 this proves the theorem.

Theorem 7-17. If A is a real n by n matrix then $[[A]]_2 = \sqrt{\lambda_m^*}$.

Proof. If A is non-singular the proof is a direct result of Theorem 7-14 and Lemma 7-16.

If A is singular, AA^T is singular and $\lambda_m^* = 0$.

Theorem 7-18. Let $A: R \to R$ be a bounded, linear function and R a normed linear space. Then for any complex scalar c, $[[cA]] = |c|\ [[A]]$.

Proof.

$$
\begin{aligned}
[[cA]] &= \inf [\ ||cAx|| : ||x|| = 1\ ] \\
&= \inf [\ |c|\ ||Ax|| : ||x|| = 1\ ] \\
&= |c| \inf [\ ||Ax|| : ||x|| = 1\ ] \\
&= |c|\ [[A]].
\end{aligned}
$$

Theorem 7-19. Let $A:R \to R$ be a linear function and R a finite dimensional normed linear space. $[[A]] = 0$ if and only if A is singular.

Proof. If A is singular, it is clear that $[[A]] = 0$ since there exists at least one non-zero element x in R such that $Ax = 0x = 0$.

If A is non-singular $[[A]] = \dfrac{1}{||A^{-1}||} \neq 0$.

Theorem 7-20. Let $A:R \to R$ and $B:R \to R$ be linear functions and R a finite dimensional, complete, normed linear space. Then $[[AB]] \geq [[A]][[B]]$.

Proof. If A or B is singular, the result is immediate. If A and B are non-singular,

$$[[AB]] = \frac{1}{||(AB)^{-1}||} = \frac{1}{||B^{-1}A^{-1}||} \geq \frac{1}{||B^{-1}||\ ||A^{-1}||} = [[B]][[A]].$$

Remark. By a suitable choice of matrices A and B it can be seen that neither $[[A + B]] \geq [[A]] + [[B]]$ nor $[[A + B]] \leq [[A]] + [[B]]$ hold in general.

Theorem 7-21. Let $S < T$ be complete subsets of a normed linear space R. Let $A:S$ onto T be a bounded, linear function with $[[A]] > 1$. Then there is a unique element x in S such that $Ax = x$.

Proof. This result follows from Theorem 2-3 and the fact that A is an expanding mapping.

Theorem 7-22. Let $R < S < T$ be complete subsets of a finite dimensional, normed linear space. Let A:R onto S and B:S onto T be bounded linear functions with $[[A]] > 1$ and $[[B]] > 1$. Then there is a unique element x in R such that $BAx = x$.

Proof. By Theorem 7-20, $[[BA]] > 1$. The result now follows directly from Theorem 7-21.

Remark. Many results analogous to those for contraction mappings can now be obtained for expanding mappings.

CHAPTER VIII

MAPPINGS IN EUCLIDEAN SPACES

Let f_k, for $1 \leq k \leq n$, be a mapping of E_n into E_1 and let

$$y_1 = f_1(x_1,\ldots,x_n)$$
$$y_2 = f_2(x_1,\ldots,x_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_n = f_n(x_1,\ldots,x_n)$$

for each element $x = (x_1,\ldots,x_n)$ in E_n. Let $F = (f_1,\ldots,f_n)$. Then $F(E_n) < E_n$. Choose x^1 and x^2 arbitrarily in E_n and let $F(x^1) = F(x_1^1,\ldots,x_n^1) = y^1 = (y_1^1,\ldots,y_n^1)$, and $F(x^2) = F(x_1^2,\ldots,x_n^2)$. If $F \in C'(E_n)$,

$$y_j^1 - y_j^2 = \sum_{k=1}^{n} D_k f_j(Z_j)(x_k^1 - x_k^2) \text{ for } j = 1,2,\ldots,n,$$

where Z_j is some vector on the line segment $L(x^1,x^2)$ connecting x^1 and x^2. Thus

$$|y_j^1 - y_j^2| \leq \sum_{k=1}^{n} |D_k f_j(Z_j)|\ |x_k^1 - x_k^2| \text{ for } 1 \leq j \leq n.$$

Summing over j,

$$\sum_{j=1}^{n} |y_j^1 - y_j^2| \leq \sum_{j=1}^{n}\sum_{k=1}^{n} |D_k f_j(Z_j)|\ |x_k^1 - x_k^2|$$

$$= \sum_{k=1}^{n} |x_k^1 - x_k^2| \sum_{j=1}^{n} |D_k f_j(Z_j)|.$$

If $\max_k \sum_{j=1}^{n} |D_k f_j(Z)| \leq \alpha < 1$ for all vectors Z in E_n, F is a contraction mapping under the l_1^n norm.

Define the Jacobian Matrix J_F so that

$$J_F(Z) = \begin{bmatrix} D_1f_1(Z) & D_2f_1(Z) & \dots & D_nf_1(Z) \\ D_1f_2(Z) & D_2f_2(Z) & \dots & D_nf_2(Z) \\ \dots & \dots & \dots & \dots \\ D_1f_n(Z) & D_2f_n(Z) & \dots & D_nf_n(Z) \end{bmatrix}$$

for all Z in E_n. Then $F:E_n \to E_n$ will be a contraction mapping if $||J_F(Z)||_1 \leq \alpha < 1$ for all vectors Z in E_n. Application of Theorem 1-4 yields the following.

<u>Theorem 8-1</u>. Let $F:E_n \to E_n$ be a vector valued function such that $F \in C'(E_n)$. If $||J_F(Z)||_1 \leq \alpha < 1$ for all Z in E_n, there exists a unique element x in E_n such that $F(x) = x$. The fixed point x is the limit of the sequence $[x_n]$ where x_1 is any element in E_n and $x_n = F(x_{n-1})$ for $n \geq 1$.

<u>Example 8-2</u>. Consider the system of equations,

$$f_1(x_1,x_2,x_3) = 0$$
$$f_2(x_1,x_2,x_3) = 0$$
$$f_3(x_1,x_2,x_3) = 0$$

where $f_k \in C'(E_n)$ for $1 \leq k \leq 3$. Let $G = (g_1,g_2,g_3)$ where

$$g_1(x_1,x_2,x_3) = f_1(x_1,x_2,x_3) + x_1$$
$$g_2(x_1,x_2,x_3) = f_2(x_1,x_2,x_3) + x_2$$
$$g_3(x_1,x_2,x_3) = f_3(x_1,x_2,x_3) + x_3$$

Then $J_G(Z) = (D_jf_i(Z) + \delta_{ij})$ and by Theorem 8-1, if $||J_G(Z)||_1 \leq \alpha < 1$

for all Z in E_3, there is a unique element x in E_3 such that $f_i(x) = 0$ for $i = 1,2,3$. This fixed point x may be obtained by the iterative process mentioned previously.

We now exhibit a similar result for expanding mappings. Returning to the proof of Theorem 8-1,

$$y_j^1 - y_j^2 = \sum_{k=1}^{n} D_k f_j(Z_j)(x_k^1 - x_k^2) \text{ for } j = 1,2,\ldots,n$$

where $Z_j \in L(x^1, x^2)$. Let $y = (y_1^1 - y_2^2, \ldots, y_n^1 - y_n^2)^T$, $x = (x_1^1 - x_1^2, \ldots, x_n^1 - x_n^2)^T$, and

$$J_F(Z_1,\ldots,Z_n) = \begin{bmatrix} D_1f_1(Z_1) & D_2f_1(Z_1) & \ldots & D_nf_1(Z_1) \\ D_1f_2(Z_2) & D_2f_2(Z_2) & \ldots & D_nf_2(Z_2) \\ \ldots & \ldots & \ldots & \ldots \\ D_1f_n(Z_n) & D_2f_n(Z_n) & \ldots & D_nf_n(Z_n) \end{bmatrix}$$

Then, in vector notation, $y = J_F(Z_1,\ldots,Z_n)\, x$. Thus $||y|| \geq [[J_F]]\ ||x||$ where

$$[[J_F]] = \inf\ [\ ||J(Z_1,\ldots,Z_n)\, x||\ :\ ||x|| = 1 \text{ and } Z_k \in E_n\ ].$$

Combining the above with Theorem 2-3 yields the following.

Theorem 8-3. Let $F: E_n$ onto E_n be a mapping where $F \in C'(E_n)$ and $[[J_F]] > 1$. Then there exists a unique point x in E_n such that $F(x) = x$.

Theorem 8-4. Let f be an analytic function of a complex variable on an open set D in the complex plane. Let C be a compact, connected subset of D so that $|f'(z)| < 1$ for all points z in C. Then, if $f(D) < D$, there exists a unique point z in D such that $f(z) = z$.

Proof. Since C is compact there exists an $\alpha < 1$ such that $|f'(z)| < \alpha$ for all z in C. Consider the open covering of C given by $[N(z,r)]$ where $|f'(z)| < \alpha$ in $N(z,2r)$. Since C is compact there exists a finite subcovering $[N(z_i,r_i)]$, $i = 1,\dots,n$. Let $\varepsilon = \min [r_i : 1 \leq i \leq n]$. Then for all z_1,z_2 in C such that $|z_1 - z_2| < \varepsilon$,

$$|f(z_1) - f(z_2)| = \left|\int_{z_1}^{z_2} f'(z)dz\right|$$

$$\leq \int_{z_2}^{z_1} |f'(z)|dz$$

$$\leq \alpha |z_1 - z_2|.$$

This implies that $f:C \to C$ is an (ε,α)-uniformly locally contraction mapping and, by Theorem 1-14, there exists one and only one point $\dot{z}$ in C such that $f(\dot{z}) = \dot{z}$.

Remark. Theorem 8-4 is given by Edelstein in [1].

CHAPTER IX

INTEGRAL EQUATIONS

Definition 9-1. Let $C[a,b]$ denote the set of all real valued functions, defined and continuous on the real interval $[a,b]$.

Definition 9-2. Let the function f be in $C[a,b]$. Then the norm of f, written $||f||_c$, is defined to be the $\sup[\ |f(t)|: a \leq t \leq b\ ]$.

Remark. It is well known that Definition 9-2 is valid and that $(C[a,b], ||\ \ ||_c)$ is a complete, normed linear space.

Theorem 9-3. Let y be an element of $C[a,b]$, K an element of $C[[a,b] \times [a,b]]$, and λ a real parameter. The integral equation

$$x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt$$

has a unique solution in $C[a,b]$ whenever $|\lambda|M(b-a) < 1$, where M denotes $\max_{s,t} |K(s,t)|$ on $[a,b] \times [a,b]$.

Proof. Let $F(x_k) = y(s) + \lambda \int_a^b K(s,t)x_k(t)dt$ for any two elements x_1 and x_2 in $C[a,b]$. Clearly $F(x_1)$ and $F(x_2)$ are elements of $C[a,b]$ and

$$F(x_1) - F(x_2) = \lambda \int_a^b K(s,t)(x_1(t) - x_2(t))dt.$$

$$|F(x_1) - F(x_2)| \leq |\lambda| \int_a^b |K(s,t)|\ |x_1(t) - x_2(t)|dt$$

$$\leq |\lambda|\ ||x_1 - x_2||_c M(b - a).$$

Thus $||F(x_1) - F(x_2)||_c \leq |\lambda|M(b - a)||x_1 - x_2||_c$. If $|\lambda|M(b - a) \leq \alpha < 1$, then $F: C[a,b] \to C[a,b]$ is a contraction mapping. Theorem 1-4 guarantees a unique x in $C[a,b]$ such that $F(x) = x$.

Remark. The proof of Theorem 9-3 implies that the fixed point x is the limit of the sequence $[x_n]$ where x_0 is any element in $C[a,b]$ and

$$x_n(s) = y(s) + \lambda \int_a^b K(s,t)x_{n-1}(t)dt \text{ for } n \geq 1.$$

Theorem 9-4. Let y be an element of $C[a,b]$ and K an element of $C[[a,b] \times [a,b]]$. The integral equation

$$x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt$$

has a unique solution in $C[a,b]$ for all real values of λ.

Proof. Let

$$Ax_1(s) = y(s) + \lambda \int_a^b K(s,t)x_1(t)dt$$

and

$$Ax_2(s) = y(s) + \lambda \int_a^b K(s,t)x_2(t)dt$$

for any x_1 and x_2 in $C[a,b]$. Then

$$Ax_1(s) - Ax_2(s) = \lambda \int_a^b K(s,t)(x_1(t) - x_2(t))dt.$$

$$|Ax_1 - Ax_2| \leq |\lambda| \int_a^b |K(s,t)| \; |x_1(t) - x_2(t)|dt$$

$$\leq |\lambda|M||x_1 - x_2||_c(s - a)$$

where $M = \max_{s,t} |K(s,t)|$ for all s,t in $[a,b] \times [a,b]$.

Similarly we find $|A^2x_1 - A^2x_2| \leq |\lambda|^2M^2||x_1 - x_2||_c\frac{(s - a)^2}{2!}$

and in general $|A^nx_1 - A^nx_2||_c \leq |\lambda|^nM^n\frac{(s - a)^n}{n!}||x_1 - x_2||_c$.

Hence

$$||A^nx_1 - A^nx_2||_c \leq |\lambda|^nM^n\frac{(s - a)^n}{n!}\ ||x_1 - x_2||_c.$$

$|\lambda|^nM^n\frac{(s - a)^n}{n!} \leq |\lambda|^nM^n\frac{(b - a)^n}{n!} < 1$ for large values of n implies

that $A^n: C[a,b] \to C[a,b]$ is a contraction mapping for some integer $n \geq 1$. By Theorem 1-5 there exists a unique element x in C[a,b] such that $Ax = x$.

Remark. According to Corollary 1-6 the fixed point x in Theorem 9-4 is the limit of the sequence $[x_n]$ where x_0 is any element in C[a,b] and

$$x_n(s) = y(s) + \lambda \int_a^s K(s,t)x_{n-1}(t)dt \text{ for } n \geq 1.$$

Definition 9-5. Let $L_2[a,b]$ denote the set of all real valued, Lebesgue measurable functions f on [a,b] for which the Lebesgue integral $\int_a^b |f(x)|^2dx$ is finite.

Definition 9-6. Let $||f||_2^* = [\int_a^b |f(x)|^2dx]^{\frac{1}{2}}$ denote the norm of an function f in $L_2[a,b]$.

Remark. It is well known that Definition 9-6 is valid and that $(L_2[a,b], ||\ \ ||_2^*)$ is a complete normed linear space.

Definition 9-7. Let $C_2[S]$ denote the set of all real valued, Lebesgue measurable functions K on S = [a,b] x [a,b] for which the

$$\int_S |K(s,t)|^2 da$$

is finite.

Definition 9-8. Let $|K|_2 = [\int_S |K(s,t)|^2 da]^{\frac{1}{2}}$ denote the norm of K when K is in $C_2[S]$.

Remark. It is well known that Definition 9-8 is valid and that $(C_2[S], |\ \ |_2)$ is a complete normed linear space.

Theorem 9-9. Let y be in $L_2[a,b]$, K in $C_2[[a,b] \times [a,b]]$ and λ a real parameter. The integral equation,

$$x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt$$

has a unique solution in $L_2[a,b]$ if $|\lambda|\ |K|_2 < 1$.

Proof. Let x_1 and x_2 be any elements of $L_2[a,b]$ and let $F(x_i) = y(s) + \lambda \int_a^b K(s,t)x_i(t)dt$ for $i = 1,2$. Clearly $F(x_1)$ and $F(x_2)$ are in $L_2[a,b]$. Since

$$F(x_1) - F(x_2) = \lambda \int_a^b K(s,t)(x_1(t) - x_2(t))dt,$$

$$\begin{aligned}||F(x_1) - F(x_2)||_2^* &= |\lambda| [\int_a^b | \int_a^b K(s,t)(x_1(t) - x_2(t))dt|^2 ds]^{\frac{1}{2}} \\ &\le |\lambda|\ [\int_a^b [\int_a^b |K(s,t)|\ |x_1(t) - x_2(t)|dt]^2 ds]^{\frac{1}{2}} \\ &\le |\lambda| (\int_a^b (\int_a^b |K(s,t)|^2 dt)ds)^{\frac{1}{2}} ||x_1 - x_2||_2 \\ &= |\lambda|\ |K|_2 ||x_1 - x_2||_2.\end{aligned}$$

If $|\lambda|\ |K|_2 < 1$, $F: L_2[a,b] \to L_2[a,b]$ is a contraction mapping. The results of Theorem 1-4 yield a unique element x in $L_2[a,b]$ such that $x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt$.

CHAPTER X

MISCELLANEOUS APPLICATIONS

Let $f: E_1 \to E_1$ be a function such that $f''(x)$ exists for all x in E_1 and $f'(x) \neq 0$ for any x in E_1. Let $g: E_1 \to E_1$ be the function

$$g(x) = x - \frac{f(x)}{f'(x)} \text{ for x in } E_1.$$

g' exists and

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2} \text{ for each x in } E_1.$$

If $\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| \leq \alpha < 1$ for all x in E_1, then by the law- of- mean, $|g(x) - g(y)| \leq \alpha\, |x - y|$ and $g: E_1 \to E_1$ is a contraction mapping. By Theorem 1-4, there exists a unique x in E_1 such that $g(x) = x - \frac{f(x)}{f'(x)} = x$. This implies the existence of a unique solution for the equation $f(x) = 0$.

The solution is the limit of the sequence $[x_n]$ where x_0 is any element in E_1 and $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$ for $n \geq 1$. This discussion proves the following theorem.

<u>Theorem 10-1</u>. Let $f: E_1 \to E_1$ be a function for which $f''(x)$ exists for all x in E_1, $f'(x) \neq 0$ for any x in E_1, and

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| \leq \alpha < 1$$

for all x in E_1. Then there is a unique point x in E_1 such that $f(x) = 0$. This point x is the limit of the sequence $[x_n]$ where x_0 is any element in E_1 and

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{for } n \geq 1.$$

Theorem 10-2. Let $f: E_1 \to E_1$ be a function for which f'' exists on E_1 and $f'(x) \neq 0$ for any x in E_1. Let

$$g(x) = x - \frac{f(x)}{f'(x)}$$

for each x in E_1. If $g(E_1) = E_1$ and $\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| \geq \alpha > 1$ for all x in E_1, there exists a unique x in E_1 such that $f(x) = 0$.

Proof. The proof of this theorem is similar to the proof of Theorem 10 - 1 but utilizes the results obtained for expanding mappings instead of those for contraction mappings. It will be omitted.

Theorem 10-3. Let $[c,d] < [a,b] < E_1$ and let $f:[a,b] \to [c,d]$ be a mapping for which $|f'(x)| < 1$ for all x in $[a,b]$. Then there exists a unique x in $[a,b]$ such that $f(x) = x$. The fixed point x is given as the limit of the sequence $[x_n]$ where x_0 is any element in $[a,b]$ and $x_n = f(x_{n-1})$ for $n \geq 1$.

Proof. This is an easy consequence of the mean value theorem and Theorem 1-4.

Remark. A result analogous to Theorem 10-3 is easily obtained for expanding mappings.

To complete this chapter, two examples will be given. They illustrate further the versatility and usefulness of contraction maps.

Example 10-4. Calculation of the limit of the sequence

$$[(2)^{\frac{1}{2}}, (2 + (2)^{\frac{1}{2}})^{\frac{1}{2}}, (2 + (2)^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}, \ldots].$$

Let $g: [0,\infty) \to [0,\infty)$ be the function given by $g(x) = (2 + x)^{\frac{1}{2}}$ for $x \geq 0$.

$$|g'(x)| = 2\left|\frac{1}{\sqrt{(2+x)}}\right| < \frac{1}{2}.$$

Thus by a direct application of the mean value theorem, g is a contraction mapping on $[0,\infty)$. By Theorem 1-4 there is a unique x in $[0,\infty)$ such that $x = \sqrt{(2 + x)}$. This x is also the limit of the sequence $[x_n]$ where $x_0 = 0$ and $x_n = \sqrt{(2 + x_{n-1})}$ for $n \geq 1$. The combination of these facts implies that the limit of the sequence is 2.

Example 10-5. The value of the continued fraction

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}.$$

Let $f: [0,\infty) \to [0,\infty)$ be the function given by $f(x) = \frac{1}{2 + x}$ for $x \geq 0$.

$$|f'(x)| = \left|\frac{1}{(2 + x)^2}\right| \leq \frac{1}{4} \text{ for } x \geq 0.$$

As in Example 11-4, f is a contraction mapping on $[0,\infty)$. Hence, there is a unique x in $[0,\infty)$ such that $x = \frac{1}{2 + x}$. x is also the limit of the sequence $[x_n]$ where $x_0 = 0$ and $x_n = \frac{1}{2 + x_{n-1}}$ for $n \geq 1$. This implies that $x = \sqrt{2} - 1$.

BIBLIOGRAPHY

1. Edelstein, M., "An Extension of Banach's Contraction Principle," Proceedings of the American Mathematical Society, vol. 12, February 1961, pp. 7-10.

2. Bers, L., Topology, New York: New York University Institute of Mathematical Sciences, 1956, p.83.

3. Hall, D. W. and Spencer, G. L. II, Elementary Topology, New York: John Wiley and Sons, Inc., 1957, p. 75.

4. Freudenthal, H. and Hurewicz, W., "Dehnungen, Verkurzungen, Isometrien," Fundamenta Mathematicae, vol. 26, 1936, pp.121-123.

5. Edrei, Albert, "On Mappings Which Do Not Increase Small Distances," Proceedings of the London Mathematical Society, (3) vol. 2, 1952, pp. 272-278.

6. Brown, T. A. and Comfort, W. W., "New Methods for Expansion and Contraction Maps in Uniform Spaces," Proceedings of the American Mathematical Society, vol. 11, 1960, pp. 483-486.

7. Williams, R. F., "Local Contractions of Compact Metric Sets Which are not Local Isometries," Proceedings of the American Mathematical Society, vol. 5, 1954, pp. 652-654.

8. Kolmogorov, A. N. and Fomin, S. V., Elements of the Theory of Functions and Functional Analysis, Boron, L. F., Rochester, New York: Graylock Press, 1954, Vol. 1, pp. 97 ff.

9. Faddeeva, V. N., Computational Methods of Linear Algebra, Benster, C. D., New York: Dover Publications, Inc., p.54 ff.